Instability of the equilibrium of a liquid below its vapour between horizontal heated plates

By ADAM HUANG AND DANIEL D. JOSEPH

Department of Aerospace Engineering and Mechanics, University of Minnesota,
110 Union St S.E., Minneapolis MN 55455, USA

We study the stability of a motionless liquid below its vapour between heated horizontal plates. The temperature of the bottom plate is held below the vaporization temperature and the top plate is hotter than the vaporization temperature. A water film is on the cold plate and a vapour film on the hot plate. We find a basic solution depending only on the variable y normal to the plates, with steady distributions of temperature, a null velocity and no phase change. The linear stability of this basic state is studied in the frame of incompressible fluid dynamics, without convection, but allowing for phase change. An ambiguity in the choice of the conditions to be required of the temperature at the phase-change boundary is identified and discussed. It is shown that the basic state of equilibrium is overstable under conditions of large temperature gradient, when the other parameters have suitable values. An analysis of the energy of the most dangerous disturbance shows that the source of the instability is associated with change of phase.

1. Introduction

There have been many studies of the stability problem for laminar film condensation or vaporization (Nusselt's solution) on an inclined cool or hot plate (see Unsal & Thomas 1980 for a fairly thorough review of this literature) and of a falling film of liquid along an inclined plate, which was decisively analysed by Yih (1955, 1963) and Benjamin (1957). Yih's problem has been generalized to a two-fluid channel flow problem for horizontal and vertical cases (see Blennerhassett 1980; Renardy 1987). Problems of the Yih type have fully developed basic flows but no phase change, and a temperature equation is not needed. Laminar film condensation is a phase-change problem, but it is posed in a semi-infinite region which excludes the existence of a fully developed steady flow; in the basic flow there is a continuous change of phase at the interface, so that the stability analysis of this problem is difficult and is treated in a quasi-steady approximation. In many of the film condensation studies the vapour phase was inactive.

The paper of Busse & Schubert (1971) was the first to address the problem of the influence of phase change on thermal convection. They generalized the usual Rayleigh–Bénard problem by including a contribution of the latent heat from phase transformation in the energy balance at the interface. They restricted their consideration to the case when the viscosity and thermal conductivity of the two phases are the same and assumed that the difference in density between the two phases is small relative to the mean density. In fact, the densities are implicitly taken to be equal across the interface in the mass balance equation so that the normal component of velocity across the interface is continuous, consistent with our (2.4)

when either $\rho_1 = \rho_2$ or the interface is stationary (2.7). Busse & Schubert assumed exchange of stability in their problem. In our problem this assumption is not correct and the loss of stability occurs always as overstability. Nevertheless their analysis showed that the fluid layer can be unstable even when the less dense phase lies above.

A number of papers have appeared in the last 20 years on the problem of mantle convection with phase change and related problems. The literature of this topic is cited in the list of references in Sotin & Parmentier (1989). All but one of these cited papers restricted their considerations to the case of small density differences in which the term $(\rho_1 - \rho_2)\,\boldsymbol{u}_\Sigma \cdot \boldsymbol{n}$ in our (2.4) may be thought to be negligible. The one exception is the paper on geothermal systems, treated as a porous media problem, by Schubert & Straus (1980) in which the density difference is large and (2.4) is not simplified. The goal of all the authors was to find the effects of phase change on convection.

The goal of this work is different; we are interested in constructing a relatively more rigorous analysis without making the assumptions which are used either in the film condensation studies or in the geophysical phase-change literature. Here the phase-change mechanism is isolated, by putting the buoyant terms of the Oberbeck–Boussinesq equation to zero. This procedure has been used frequently before in the literature on film condensation. In the course of our study we came to realize that the correct conditions on the temperature at a phase-change boundary are not known. There are different possibilities. If we require that the temperature at a phase boundary is in thermodynamic equilibrium, then the saturation temperature is uniquely determined by the pressure through the Clapeyron relation. If the pressure on the water side is different than on the vapour side, as in our stability study, the temperature will be discontinuous. We can choose only two conditions for the temperature at a phase change boundary, so the thermodynamic equilibrium (the Clapeyron relation) excludes thermal equilibrium (continuity of temperature). This issue seems not to have been addressed in the flm condensation literature or the geophysical phase-change literature, where it is conventional to require thermal equilibrium between water and its vapour together with thermo-dynamic equilibrium for the vapour, but not the water; that is, the temperature of the water is at the saturation value appropriate to pressure in the vapour (see Plesset & Zwick 1954; Gebhart 1971; Ishii 1975). In every study of this problem known to us the two phases are required to be in thermal equilibrium, the temperature must be continuous across the phase-change interface. But temperature discontinuities cannot be eliminated by rigorous application of first principles. Schrage (1953) says 'There is no reason why the temperature of the gas phase should necessarily be the same as that of the liquid or solid surface in all cases.' Indeed, classic kinetic theory calculations (see Hirschfelder, Curtis & Bird 1954; Pao 1971; Aoki & Cercignani 1983; Onishi 1984 and Cercignani, Fiszdon & Frezzotti 1985) indicate that for a monatomic vapour large temperature jumps exist at interfaces. Shankar & Deshpande (1990) have measured the temperature distribution in the vapour between an evaporating liquid surface and a cooler condensing surface in water, Freon 113 and mercury. The temperature profiles obtained in mercury showed jumps at the interface as large as almost 50 % of the applied temperature difference.

In the present study we allow phase change, and the vapour phase is active. The second wall allows the system to attain a steady fully developed basic state. We are not in the frame of laminar film condensation because phase changes do not occur in the basic flow. We carried out calculations for three cases: when the water and va, our are at saturation values corresponding to their own pressures (thermo-dynamic ̴quilibrium) and when the vapour or the water is at saturation and the

temperature is continuous (thermal equilibrium). The results of the stability analysis do not depend strongly on the choice of the three aforementioned conditions for the temperature at the interface in the case of water and water vapour.

2. Governing equations and basic solution

We assume that physical properties of the fluids, i.e. the viscosity, density and conductivity, are constants independent of pressure and temperature, as is usually done in film condensation problems. This means that the pressure will be a dynamical variable uncoupled from thermodynamics and that convective currents set up by the action of gravity on thermally induced variations of density will be ignored.

We consider two-dimensional flow, and its physical configuration is shown in figure 1. The velocities of both phases satisfy the incompressible Navier–Stokes equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{2.1}$$

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{u}. \tag{2.2}$$

In the temperature equation we shall neglect the dissipation term:

$$\rho C_p \frac{\mathrm{d}T}{\mathrm{d}t} = k \nabla^2 T. \tag{2.3}$$

At the interface $I(x,t) = y - H(x,t) = 0$ we have mass, momentum and energy balances (see Joseph 1990) (in what follows subscript $i = 1$ or 2 refers to vapour or water respectively):

mass: $$-\dot{m}(x,t) = \rho_1(\boldsymbol{u}_1 - \boldsymbol{u}_\Sigma) \cdot \boldsymbol{n} = \rho_2(\boldsymbol{u}_2 - \boldsymbol{u}_\Sigma) \cdot \boldsymbol{n}, \tag{2.4}$$

momentum: $$\dot{m}[\![\boldsymbol{u}]\!] - [\![p]\!]\boldsymbol{n} + 2[\![\mu \mathrm{D}[\boldsymbol{u}] \cdot \boldsymbol{n}]\!] = 2\tilde{H}\sigma \boldsymbol{n}, \tag{2.5}$$

energy: $$-[\![k\boldsymbol{\nabla}T]\!] \cdot \boldsymbol{n} = 2[\![\mu(\boldsymbol{u} - \boldsymbol{u}_\Sigma) \cdot \mathrm{D}[\boldsymbol{u}] \cdot \boldsymbol{n}]\!] + \dot{m}[\![h + \tfrac{1}{2}|\boldsymbol{u} - \boldsymbol{u}_\Sigma|^2]\!], \tag{2.6}$$

where $\tilde{H}$ is the mean curvature of the interface, σ is the surface tension coefficient, and $\boldsymbol{n}$ is the normal of the interface directed from vapour to water, $[\![\cdot]\!]$ is defined as $(\cdot)_1 - (\cdot)_2$. $\boldsymbol{u}_\Sigma$ can be regarded as the interface velocity, which satisfies

$$\boldsymbol{u}_\Sigma \cdot \boldsymbol{n} = \frac{\partial I / \partial t}{|\boldsymbol{\nabla} I|}. \tag{2.7}$$

In general the fluid velocity $\boldsymbol{u} \neq \boldsymbol{u}_\Sigma$, because of mass accretion or depletion, but we may assume that the interface and fluid have the same tangential component of velocity, $(\boldsymbol{u} - \boldsymbol{u}_\Sigma) \cdot \boldsymbol{t} = 0$ on the interface. Then

$$[\![\boldsymbol{u}]\!] \cdot \boldsymbol{t} = 0, \tag{2.8}$$

where $\boldsymbol{t}$ is tangent on the interface.

As we have mentioned before we do not know what condition the temperature should satisfy at the interface. In this study we shall consider three different choices of the interfacial temperature conditions, referred as cases I, II, III, where T_s is the vaporization temperature:

Case I. Thermodynamic equilibrium of both phases

$$T_{si} = T(P_i). \tag{2.9}$$

Case II. Thermal equilibrium and thermodynamic equilibrium of vapour

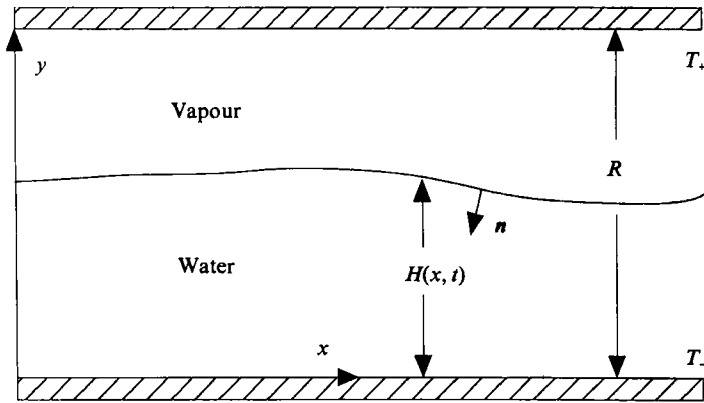$$[\![T]\!] = 0, \quad T_1 = T(P_1). \tag{2.10}$$

FIGURE 1. The physical configuration of this two-phase flow problem.

Case III. Thermal equilibrium and thermodynamic equilibrium of water

$$[T] = 0, \quad T_2 = T(P_2), \tag{2.11}$$

and compare the results.

Boundary conditions are

at $y = 0$, $\quad\quad\quad\quad u_2 = v_2 = 0, \quad T_2 = T_-,$ $\quad\quad\quad\quad$ (2.12)

at $y = R$, $\quad\quad\quad\quad u_1 = v_1 = 0, \quad T_1 = T_+.$ $\quad\quad\quad\quad$ (2.13)

There is a steady, developed, motionless solution

$$(u, v, p, T, H) = (0, 0, P, T, H_0)$$

where H_0 is a constant:

$$H_0 = \frac{(T_s - T_-)\, k_2 R}{(T_+ - T_s)\, k_1 + (T_s - T_-)\, k_2}, \tag{2.14}$$

$$\left.\begin{aligned} T_1(y) &= T_+ - (T_+ - T_s)\frac{R - y}{R - H_0}, \quad y \in (H_0, R),\\[2mm] T_2(y) &= T_s - (T_s - T_-)\frac{H_0 - y}{H_0}, \quad y \in (0, H_0). \end{aligned}\right\} \tag{2.15}$$

The pressure P is just the hydrostatic pressure. At the vapour–water interface $y = H_0$, $T_1 = T_2 = T_s$, the pressure is continuous, and the basic flow is in thermodynamic equilibrium with

$$T_s = T(P(H_0)), \quad \dot{m} = 0. \tag{2.16}$$

So the different choices of interfacial temperature conditions give rise to the same basic state.

3. Perturbation equations and normal modes

To make our equations dimensionless we use following scales: length, H_0; velocity, $V_0 = \mu_1 \zeta/(\rho_1 H_0 m)$; time, H_0/V_0; pressure, $\rho_i V_0^2$, where $i = 1$ (vapour) or 2 (water); temperature, $T_s - T_-$.

The dimensionless parameters listed below will appear in the dimensionless equations:

density ratio $\qquad\qquad\quad \zeta = \rho_2/\rho_1,$

viscosity ratio $\qquad\qquad\quad m = \mu_2/\mu_1,$

Reynolds number $\qquad\quad\; \mathbb{R}_i = \rho_i V_0 H_0/\mu_i,$

Prandtl number $\qquad\qquad Pr_i = \mu_i c_i/k_i,$

Péclet number $\qquad\qquad Pe_i = \mathbb{R}_i Pr_i,$

Weber number $\qquad\qquad W = \sigma\rho_1 H_0 m^2/(\mu_1^2 \zeta^2),$

thermal conductivity ratio $\quad \xi = k_2/k_1,$

heat of vaporization number $\quad \Gamma = k_1 T_s m(1-\tau_2)/(\zeta h_{fg}\mu_1),$

dimensionless slope of
the Clapeyron curve $\qquad \Pi_i = \dfrac{\zeta^2 \rho_i \mu_1}{h_{fg} H_0^2 m^2}\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right)\dfrac{1}{1-\tau_2},$

gravity number $\qquad\qquad \tilde{G} = g\rho_1^2 H_0^3 m^2/(\mu_1^2 \zeta^2),$

hot-wall temperature ratio $\quad \tau_1 = T_+/T_s,$

cold-wall temperature ratio $\quad \tau_2 = T_-/T_s,$

relative distance between
the two plates $\qquad\qquad r = \dfrac{R}{H_0} = 1 + \dfrac{\tau_1 - 1}{\xi(1-\tau_2)}.$

If we specify the fluids as water and water vapour and take all their material parameters as their published values at $T_s = 100\ ^\circ\mathrm{C}$,

$$\begin{aligned}
\rho_1 &= 0.585\ \mathrm{g/m^3}, & \rho_2 &= 0.965\ \mathrm{g/cm^3}, \\
\mu_1 &= 0.0125\ \mathrm{cP}, & \mu_2 &= 0.28\ \mathrm{cP}, \\
k_1 &= 2.5\times 10^{-4}\ \mathrm{J/cm\ s\ ^\circ C}, & k_2 &= 6.8\times 10^{-3}\ \mathrm{J/cm\ s\ ^\circ C}, \\
C_{p1} &= 1.96\ \mathrm{J/g\ ^\circ C}, & C_{p2} &= 4.18\ \mathrm{J/g\ ^\circ C}, \\
h_{fg} &= 2.257\times 10^3\ \mathrm{J/g}, & \sigma &= 64.4\ \mathrm{dynes/cm},
\end{aligned}$$

we are left with three independent parameters: $H_0,\ r,\ \tau_2$; and

$$\zeta = 1.65\times 10^3, \quad m = 22.4, \quad \xi = 27.2,$$

are constants and

$$\tilde{G} = 4.34\times 10^6\, H_0^3, \quad W = 4.84\times 10^4\, H_0, \quad \Gamma = 4.6\times 10^{-3}(1-\tau_2),$$

$$\Pi_1 = 1.0\times 10^{-12}\, H_0^{-2}\frac{1}{1-\tau_2}, \quad \Pi_2 = 1.69\times 10^{-9}\, H_0^{-2}\frac{1}{1-\tau_2},$$

where H_0 is measured in m.

The basic temperature profile in dimensionless form is given by

$$T_1(y) = \frac{\tau_1}{1-\tau_2} - \frac{\tau_1 - 1}{1-\tau_2}\frac{r-y}{r-1}, \tag{3.1}$$

$$T_2(y) = \frac{\tau_2}{1-\tau_2} + y. \tag{3.2}$$

Let $u,\ v,\ p,\ \theta,\ h$ be perturbations of the basic state. Using the normal-mode decomposition of solutions

$$\begin{aligned}
v(x,y,t) &= \{u(y), iv(y)\}\exp(i\alpha(x-ct)), \\
p(x,y,t) &= p(y)\exp(i\alpha(x-ct)), \\
\theta(x,y,t) &= \theta(y)\exp(i\alpha(x-ct)), \\
h(x,t) &= h\exp(i\alpha(x-ct)),
\end{aligned}$$

we get the following equations for the amplitudes $u(y)$, $v(y)$, $p(y)$, $\theta(y)$ and h:

$$\alpha u + v' = 0, \tag{3.3}$$

$$-\alpha c u = -\alpha p + \frac{\mathrm{i}}{\mathbb{R}}(\alpha^2 u - u''), \tag{3.4}$$

$$-\alpha c v = p' + \frac{\mathrm{i}}{\mathbb{R}}(\alpha^2 v - v''), \tag{3.5}$$

$$\alpha c \theta - v T' = -\frac{\mathrm{i}}{Pe}(\alpha^2 \theta - \theta''), \tag{3.6}$$

together with boundary conditions

$$u_1(r) = v_1(r) = \theta_1(r) = 0, \tag{3.7}$$

and

$$u_2(0) = v_2(0) = \theta_2(0) = 0, \tag{3.8}$$

and interface conditions at $y = 1$:

$$v_1 - \zeta v_2 = \alpha c(\zeta - 1)h, \tag{3.9}$$

$$[\![u]\!] = 0, \tag{3.10}$$

$$[\![m(-\alpha v + u')]\!] = 0, \tag{3.11}$$

$$2\left[\!\left[\frac{\mathrm{i}\zeta v'}{\mathbb{R}}\right]\!\right] - [\![\zeta p]\!] - (W\alpha^2 + \tilde{G}(\zeta - 1))h = 0, \tag{3.12}$$

$$\Gamma[\![\zeta \theta']\!] - \mathrm{i}(\alpha c h + v_1) = 0. \tag{3.13}$$

Temperature conditions will be chosen from following:

$$\theta_i = \Pi_i p_i + (-T'_i - \Pi_i \tilde{G})h, \quad i = 1 \text{ or } 2, \tag{3.14}$$

$$[\![\theta]\!] + [\![\mathrm{d}T/\mathrm{d}y]\!]h = 0. \tag{3.15}$$

We can eliminate u and p from (3.3)–(3.6), and get following equations:

$$v^{(4)} - 2\alpha^2 v'' + \alpha^4 v = \mathrm{i}\alpha c \mathbb{R}(\alpha^2 v - v''), \tag{3.16}$$

$$\theta'' - \alpha^2 \theta - \mathrm{i} Pe\, v T' = -\mathrm{i}\alpha c Pe\, \theta. \tag{3.17}$$

We introduce $\hat{\alpha}$ and $\hat{\gamma}$ by defining

$$\hat{\alpha}^2 = \alpha^2 - \mathrm{i}\alpha c \mathbb{R} \quad \text{and} \quad \hat{\gamma}^2 = \alpha^2 - \mathrm{i}\alpha c\, Pe.$$

The general solution of this system must have following form:

$$\left.\begin{array}{l} v(y) = A\,\mathrm{e}^{\alpha y} + B\,\mathrm{e}^{-\alpha y} + C\,\mathrm{e}^{\hat{\alpha} y} + D\,\mathrm{e}^{-\hat{\alpha} y}, \\ \theta(y) = B\,\mathrm{e}^{\hat{\gamma} y} + F\,\mathrm{e}^{-\hat{\gamma} y} + \tilde{\theta}(y), \end{array}\right\} \tag{3.18}$$

where

$$\tilde{\theta}(y) = \frac{T'}{\alpha c}\left[A\,\mathrm{e}^{\alpha y} + B\,\mathrm{e}^{-\alpha y} + \frac{Pe}{Pe - \mathbb{R}}(C\,\mathrm{e}^{\hat{\alpha} y} + D\,\mathrm{e}^{-\hat{\alpha} y})\right].$$

Substituting these expressions into the boundary and interfacial equations forms an eigenvalue problem for a 13×13 matrix: the 13 unknowns are the coefficients A, B, C, D, E, F of the vapour and water phases and the interface position variable h. We cannot solve this matrix analytically; however, for each set of parameter, we can obtain the eigenvalues of this matrix numerically.

4. Long waves

For long-wave disturbances ($\alpha = 0$), the system is slightly different. We are able to calculate the determinant of the 13×13 matrix and obtain an explicit equation for the zeroth-order term of αc in the long-wave expansion, which agrees with the general solution (3.18).

Suppose $v_0(y), \theta_0(y), h_0$ are the zeroth-order terms in the long-wave expansion, and σ_0 is the leading order of αc in the expansion. At this order, the system becomes

$$v_0^{iv} = -i\sigma_0 \, \mathbb{R} v_0'' \tag{4.1}$$

$$\theta_0'' - iPe\, v_0 \frac{dT}{dy} = -i\sigma_0 Pe\, \theta_0. \tag{4.2}$$

This system admits a solution in the following form:

$$v_0(y) = A\, e^{i\hat{\beta}y} + B\, e^{-i\hat{\beta}y} + Cy + D, \tag{4.3}$$

$$\theta_0(y) = E\, e^{i\hat{\phi}y} + F\, e^{-i\hat{\phi}y} + \tilde{\theta}_0(y), \tag{4.4}$$

where $\hat{\beta}^2 = i\sigma_0\,\mathbb{R}$, $\hat{\phi}^2 = i\sigma_0 Pe$. Bringing this expression into the boundary and interface conditions, after some manipulating, we find that the eigenvalue σ_0 satisfies the following equation:

$$\{(\zeta m)^{\frac{1}{2}}(e^{2i\hat{\beta}_1(1-r)} - 1)(e^{2i\hat{\beta}_2} + 1) - (e^{2i\hat{\beta}_1(1-r)} + 1)(e^{2i\hat{\beta}_2} - 1)\}$$
$$\times \{\lambda^{\frac{1}{2}}(e^{2i\hat{\phi}_1(1-r)} + 1)(e^{2i\hat{\phi}_2} - 1) - (e^{2i\hat{\phi}_1(1-r)} - 1)(e^{2i\hat{\phi}_2} + 1)\} = 0, \tag{4.5}$$

where $\lambda = \zeta\xi(C_{p1}/C_{p2})$ is a constant. Note that $\sigma_0 = 1$ is a solution of (4.5) and only r appears as a parameter. We traced this zero eigenvalue numerically, increasing α, and found that the real part becomes negative and with a non-zero imaginary part. For any given r, we can solve the above equation and get all the other eigenvalues, and they are all negative. This shows that long-wave disturbances are neutrally stable at zeroth order.

5. Energy analysis

The key to understanding the predictions of the linear theory of stability is the determination of the maximum growth rate in unstable cases. This gives rise to a distinguished length and wave speed and when combined with an energy analysis of this fastest growing mode, it can even be used to forecast the flow type. The energy analysis in this problem is different than some other two-fluid problems (see Hu & Joseph 1989) because the temperature or thermal energy equation is coupled to mechanical energy through the mechanism of phase change. In this case we may get two energy balances, one for mechanical energy designated with a subscript M and another for thermal energy designated with subscript T. Suppose (u, v, θ) are the components of an eigenvector associated with the maximum growth rate; multiplying (3.4) and (3.5) by u^* and v^*, the complex conjugates of u and v respectively, and integrating the sum of them over both the liquid and vapour regions, we get

$$-\alpha c \langle (u^2 + v^2) \rangle = \left\langle \frac{i}{\mathbb{R}}\left(\alpha^2(u^2 + v^2) + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2\right)\right\rangle + \left[\!\left[\frac{\zeta i}{\mathbb{R}}(u'u^* + v'v^*)\right]\!\right] - [\![\zeta pv^*]\!], \tag{5.1}$$

where $\langle\,\rangle = \int_1 + \int_2$; $u^2 = uu^*$, $v^2 = vv^*$. The imaginary part of the left-hand side of

(5.1) represents the growth rate of the energy of the interface and the right-hand side may be split into two parts

$$\dot{E}_{\mathrm{M}} = B_{\mathrm{M}} - D_{\mathrm{M}},$$ (5.2)

where

$$\dot{E}_{\mathrm{M}} = \alpha c_i \langle (u^2 + v^2) \rangle,$$

$$D_{\mathrm{M}} = \left\langle \frac{1}{\mathbb{R}} \left(\alpha^2 (u^2 + v^2) + \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 \right) \right\rangle \quad \text{(viscous dissipation rate)},$$

$$B_{\mathrm{M}} = -\operatorname{Im} \left\{ \left[\!\!\left[\frac{\zeta i}{\mathbb{R}} (u'u^* + v'v^*) \right]\!\!\right] - [\![\zeta p v^*]\!] \right\} \quad \text{(energy production at the interface)}.$$

We may transform the last term of B_{M} as follows:

$$[\![\zeta p v^*]\!] = [\![\zeta p]\!] v_2^* + [\![v^*]\!] p_1,$$ (5.3)

where $[\![\zeta p]\!]$ can be evaluated from the jump condition (3.12) as

$$2 \left[\!\!\left[\frac{\zeta i}{\mathbb{R}} v' \right]\!\!\right] - (\alpha^2 W + \tilde{G}(\zeta - 1)) h.$$

Then B_{M} can be further decomposed into four parts:

$$B_{\mathrm{M}} = B_1 + Ph + B_2 + G,$$ (5.4)

where

$$B_1 = -\alpha^2 W \operatorname{Im}(h v_2^*)$$ (5.5)

can be regarded as the energy supply rate due to the surface tension;

$$Ph = \operatorname{Im}([\![v^*]\!] p_1) = \frac{(1 - \zeta) \Gamma}{\zeta} \operatorname{Im}(i [\![\zeta \theta']\!]^* p_1)$$ (5.6)

arises from phase change;

$$B_2 = -\operatorname{Im} \left\{ \left[\!\!\left[\frac{\zeta i}{\mathbb{R}} (u'u^* + v'v^*) \right]\!\!\right] - 2 \left[\!\!\left[\frac{\zeta i}{\mathbb{R}} v' \right]\!\!\right] v_2^* \right\}$$ (5.7)

represents interfacial friction due to the viscosity difference; and

$$G = \tilde{G}(1 - \zeta) \operatorname{Im}(h v_2^*)$$ (5.8)

is the gravity term. Then (5.2) becomes

$$\dot{E}_{\mathrm{M}} = B_1 + Ph + B_2 + G - D_{\mathrm{M}}.$$ (5.9)

Similarly we get the energy balance for the temperature disturbance as

$$-\alpha c \langle \theta^2 \rangle = \left\langle \frac{i}{Pe} \left(\alpha^2 \theta^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2 \right) \right\rangle - \left\langle v\theta^* \frac{\mathrm{d}T}{\mathrm{d}y} \right\rangle + \left[\!\!\left[\frac{i}{Pe} \theta' \theta^* \right]\!\!\right],$$ (5.10)

which can be written as

$$\dot{E}_{\mathrm{T}} = I_{\mathrm{T}} - D_{\mathrm{T}} + B_{\mathrm{T}},$$ (5.11)

where

$$\dot{E}_{\mathrm{T}} = \alpha c_i \langle \theta^2 \rangle, \qquad D_{\mathrm{T}} = \left\langle \frac{1}{Pe} \left(\alpha^2 \theta^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2 \right) \right\rangle,$$

$$I_{\mathrm{T}} = \operatorname{Im} \left\langle v\theta^* \frac{\mathrm{d}T}{\mathrm{d}y} \right\rangle, \quad B_{\mathrm{T}} = -\operatorname{Re} \left(\left[\!\!\left[\frac{1}{Pe} \theta' \theta^* \right]\!\!\right] \right).$$

6. Results and discussion

We first compared results for three different choices of the interfacial temperature. We found that the difference between water and its vapour is rather small. Table 1 displays a comparison for some typical parameter values. Case I is the case when
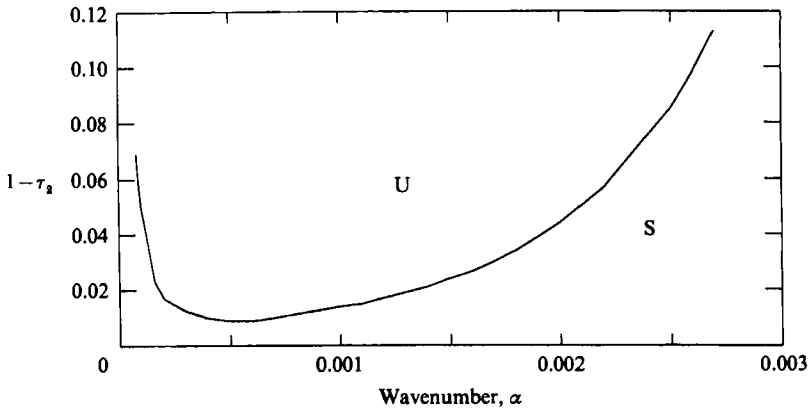
FIGURE 2. Neutral curve when $H_0 = 0.001$ m, $r = 1.3$, $\tilde{G} = 0.004\,34$, $W = 48.4$; Γ, Π_1 and Π_2 change with $(1 - \tau_2)$. U denotes unstable, S stable.

Parameters	Eigenvalue with the maximum growth rate		
	Case I	Case II	Case III
$H_0 = 0.001$ m $\alpha = 0.001$ $r = 1.10$ $\tau_2 = 0.99$	$\pm 3.2593 \times 10^{-3}$, $3.9424\mathrm{i} \times 10^{-4}$	$\pm 3.2592 \times 10^{-3}$, $3.9430\mathrm{i} \times 10^{-4}$	$\pm 3.2592 \times 10^{-3}$, $3.9424\mathrm{i} \times 10^{-4}$
$H_0 = 0.0001$ m $\alpha = 0.0005$ $r = 1.10$ $\tau_2 = 0.99$	$\pm 1.8923 \times 10^{-3}$, $-1.3569\mathrm{i} \times 10^{-4}$	$\pm 1.8916 \times 10^{-3}$, $-1.3572\mathrm{i} \times 10^{-4}$	$\pm 1.8921 \times 10^{-3}$, $-1.3569\mathrm{i} \times 10^{-4}$

TABLE 1. Comparison of the results from three different choices of interfacial temperature conditions

thermodynamic equilibrium is required for both phases, case II is when temperature continuity replaces thermodynamic equilibrium in the water, and case III is when temperature continuity replaces thermodynamic equilibrium in the vapour. The fact that the differences between the three cases are small is understandable because the parameters Π_1 and Π_2 are small. This implies that the influence of saturation pressure on the saturation temperature is negligible.

In the following we shall present results only for case II, which is the case usually treated in the literature on film condensation. We have three independent parameters: H_0, r, τ_2. First we consider the case when H_0 and r are fixed, τ_2 is the parameter, and get the neutral curve shown in figure 2. The energy analysis (tables 2, 3) shows that the instability is solely caused by the phase-change term. Note that both parameters Π_1, Γ change with τ_2; however, for this parameter range Π_1 is usually rather small. It follows that in the range of parameters under consideration Γ is the main stability parameter. This can also be seen from (5.6) which shows that the phase change term is 'directly proportional to' Γ. The flow gets more unstable when Γ is larger.

We also considered the case when H_0 and τ_2 are fixed and r varies. The result is shown as figure 3. The conclusion is more clearly seen in figure 4, which gives the
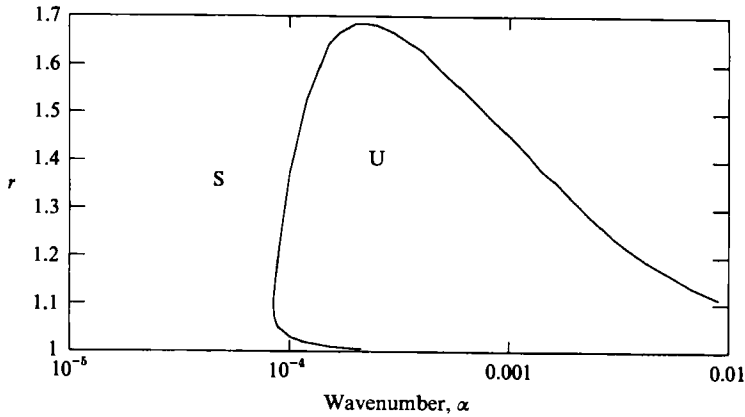
FIGURE 3. Neutral curve when τ_2 (i.e. the temperature of the lower plane) is fixed (0.95):
$H_0 = 0.001$ m, $\tilde{G} = 0.00434$, $W = 48.4$, $\Gamma = 0.231 \times 10^{-3}$, $\Pi_1 = 0.2 \times 10^{-4}$, $\Pi_2 = 0.0338$.
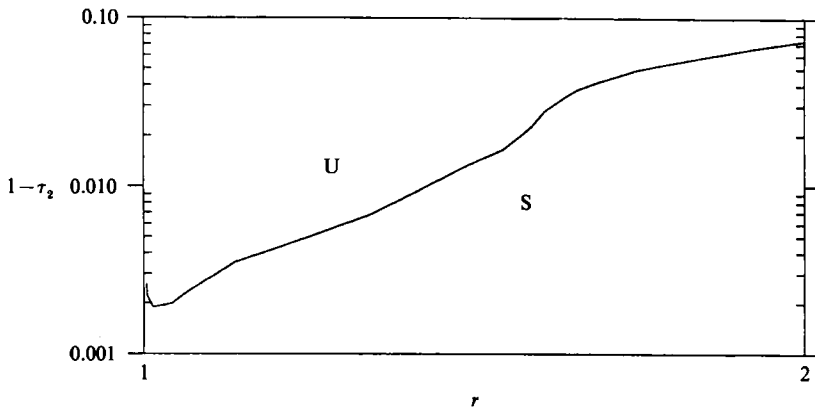

FIGURE 4. Neutral curve when $H_0 = 0.001$ m, $\tilde{G} = 0.00434$, $W = 48.4$.

neutral curve when H_0 is fixed, r and τ_2 are control parameters. Each point of the curve is a critical point. The most unstable configuration is one with a finite but very thin layer of vapour.

Very thick and very thin vapour layers are stable. This is probably a manifestation of limiting behaviour in which layers with only water and only vapour are stable. The longest waves (very small α) are always stable but the system is most unstable to long waves of finite length.

Tables 2 and 3 are results of the energy analysis of the mode with maximum growth rate. Note that the mechanical energy terms are normalized so that the viscous dissipation term $D_M = 1$, and D_T was used to normalize (5.11). The phase-change term Ph is dominant and destabilizing. The surface tension term B_1 and gravity G are stabilizing but of negligible magnitude compared with the phase-change term. The energy balance is essentially a balance between the phase-change term and the viscous dissipation. This also helps to explain the fact that the stability does not change much with H_0 (compare table 2 and table 3) when r and τ_2 are fixed because H_0 appears through W and $\tilde{G}$ which represent surface tension and gravity respectively, both of which are insignificant in the energy balance.

$1-\tau_2$	$\tilde{\alpha}$	$\tilde{\alpha}c_i$	$\dot{E}_M$	B_1	Ph
0.017	3.6×10^{-5}	1.3×10^{-6}	8.9×10^{-4}	-3.9×10^{-16}	1.00
0.030	1.1×10^{-4}	5.0×10^{-4}	1.7×10^{-1}	-3.3×10^{-13}	1.17
0.040	1.9×10^{-4}	1.0×10^{-3}	2.3×10^{-1}	-1.2×10^{-12}	1.23
0.070	5.7×10^{-4}	4.0×10^{-3}	2.8×10^{-1}	-1.1×10^{-11}	1.34

$1-\tau_2$	B_2	G	$\dot{E}_T$	B_T	I_T-D_T
0.017	1.8×10^{-5}	-4.5×10^{-6}	5.5×10^{-6}	1.0	-9.9×10^{-1}
0.030	-1.5×10^{-6}	-4.2×10^{-4}	6.9×10^{-4}	1.0	-9.9×10^{-1}
0.040	6.9×10^{-6}	-5.0×10^{-4}	8.2×10^{-4}	1.0	-9.9×10^{-1}
0.070	-3.0×10^{-6}	-5.1×10^{-4}	1.0×10^{-3}	1.0	-9.9×10^{-1}

TABLE 2. Energy analysis for the mode of maximum growth rate in the case when $H_0 = 0.01$ m, $\tilde{G} = 4.34$, $W = 484$ and $\tau_1 = 1.2$. $\tilde{\alpha}$ is the wavenumber for which the growth rate is maximum

$1-\tau_2$	$\tilde{\alpha}$	$\tilde{\alpha}c_i$	$\dot{E}_M$	B_1	Ph
0.017	3.6×10^{-4}	1.6×10^{-6}	1.2×10^{-3}	-4.6×10^{-13}	1.00
0.030	1.1×10^{-3}	5.0×10^{-4}	1.6×10^{-1}	-3.4×10^{-10}	1.16
0.040	1.8×10^{-3}	1.1×10^{-3}	2.4×10^{-1}	-1.1×10^{-9}	1.24
0.070	5.2×10^{-3}	4.0×10^{-3}	3.8×10^{-1}	-1.0×10^{-8}	1.38

$1-\tau_2$	B_2	G	$\dot{E}_T$	B_T	I_T-D_T
0.017	$+2.0 \times 10^{-5}$	-5.5×10^{-7}	7.1×10^{-6}	1.0	-9.99×10^{-1}
0.030	-1.5×10^{-6}	-4.2×10^{-5}	6.9×10^{-4}	1.0	-9.99×10^{-1}
0.040	4.6×10^{-6}	-5.2×10^{-4}	8.2×10^{-4}	1.0	-9.99×10^{-1}
0.070	$+3.5 \times 10^{-6}$	-5.6×10^{-4}	1.0×10^{-3}	1.0	-9.99×10^{-1}

TABLE 3. Same as table 2 except $H_0 = 0.001$ m, $\tilde{G} = 0.00434$, $W = 48.4$ and $\tau_1 = 1.2$

7. Conclusions

The influence on the flow stability of phase change has long been a major concern in the studies of film condensation or evaporization. However, it seems that previous studies (Unsal & Thomas 1978, 1980) were restricted to cases when the flow domain was semi-infinite. The basic flow in that case (usually Nusselt's solution) is not steady, the water film is continuously evaporating or condensing and it is found that condensation is stabilizing and vaporization is destabilizing (Unsal & Thomas 1978, 1980).

The problem of the influence of phase change on thermal convection has been considered in the literature on geophysical fluid mechanics (see Sotin & Parmentier 1989 for references), starting with the work of Busse & Schubert (1971). The assumptions of these authors, i.e. small density differences for all authors but Schubert & Straus (1980) and Darcy's law for Schubert & Straus, seem to be compatible with the assumption that the marginal eigenfunctions are steady at criticality, i.e. exchange of stability. This assumption is not correct in the problem of phase change between water and vapour, which is treated here. In our problem instability occurs with complex eigenvalues (see table 1), which means that either standing or travelling waves can be expected when the steady state loses stability. The choice between standing and travelling wave awaits further nonlinear studies of bifurcation.

The issue of interfacial temperature conditions we believe is an important one, which has recently been considered for general phase problems by Truskinovsky

(1991). The problem arises whenever a pressure jump across the interface is allowed even when the phases are solid or liquid. If the pressures on each side of the interface are different and the temperature is at saturation, then there must be a discontinuity of temperature. Different choices of thermal boundary conditions are possible. Although in the special case treated here, water and its vapour, the stability results do not depend strongly on the choice of conditions on the temperature at the interface, we do not expect weak dependence in general especially when the dependence of the saturation temperature on the pressure is not small. The choice of temperature conditions at a phase-change boundary goes beyond continuum thermomechanics and appears to require some form of molecular theory.

REFERENCES

AOKI, K. & CERCIGNANI, C. 1983 Evaporation and condensation on two parallel plates at finite Reynolds numbers. *Phys. Fluids* **26**, 1163.

BENJAMIN, T. B. 1957 Wave formation in laminar flow down an inclined plane. *J. Fluid Mech.* **2**, 554–574.

BLENNERHASSETT, P. J. 1980 On the generation of waves by wind. *Proc. R. Soc. Lond.* A **298**, 451–494.

BUSSE, F. H. & SCHUBERT, G. 1971 Convection in a fluid with two phases. *J. Fluid Mech.* **46**, 801–812.

CERCIGNANI, C., FISZDON, W. & FREZZOTTI, A. 1985 The paradox of the inverted temperature profiles between an evaporating and a condensing surface. *Phys. Fluids* **28**, 237.

GEBHART, B. 1971 *Heat Transfer*, 2nd edn. McGraw-Hill.

HIRSCHFELDER, J. V., CURTIS, C. F. & BIRD, R. B. 1954 *Molecular Theory of Gases and Liquids.* John Wiley and Sons.

HU, H. & JOSEPH, D. D. 1989 Lubricated pipelining: stability of core–annular flow. Part 2. *J. Fluid Mech.* **205**, 359.

ISHII, M. 1975 *Thermo-Fluid Dynamic Theory of Two-phase Flow.* Eyrolles.

JOSEPH, D. D. 1990 *Fluid Dynamics of Viscoelastic Liquids.* Springer.

ONISHI, Y. 1984 *14th Intl Symp. on Rarefied Gas Dynamics, Tsukuba, Japan.*

PAO, Y. P. 1971 Application of kinetic theory to the problem of evaporation and condensation. *Phys. Fluids* **14**, 306.

PLESSET, M. S. & ZWICK, S. A. 1954 The growth of vapor bubbles in superheated liquids. *J. Appl. Phys.* **25**, 493.

RENARDY, Y. 1987 Viscosity and density stratification in vertical Poiseuille flow. *Phys. Fluids* **30**, 1658.

SCHRAGE, R. 1953 *A Theoretical Study of Interphase Mass Transfer.* Columbia University Press.

SCHUBERT, G. & STRAUS, J. 1980 Gravitational stability of water over steam in vapor-dominated geothermal systems. *J. Geophys. Res.* **85**, 6505.

SHANKAR, P. N. & DESHPANDE, M. D. 1990 On the temperature distribution in liquid-vapor phase change between plane liquid surfaces. *Phys. Fluids* A **2**, 1030.

SOTIN, C. & PARMENTIER, E. M. 1989 On the stability of a fluid layer containing a univariant phase transition: application to planetary interiors. *Phys. Earth Planet. Inter.* **55**, 10–25.

TRUSKINOVSKY, L. 1991 Kinks versus shocks. In *Shock Induced Transitions and Phase Structures in General Media* (ed. R. Fosdick, E. Dunn, M. Slemrod). IMA vol. 36, Springer (to appear).

UNSAL, M. & THOMAS, W. C. 1978 Linear stability analysis of film condensation. *Trans. ASME* C: *J. Heat Transfer* **100**, 629.

UNSAL, M. & THOMAS, W. C. 1980 Nonlinear stability analysis of film condensation. *Trans. ASME C: J. Heat Transfer* **102**, 483.

YIH, C. S. 1955 Stability of parallel laminar flow with a free surface. In *Proc. Second US Natl Congr. of Applied Mechanics*, pp. 623. ASME.

YIH, C. S. 1963 Stability of liquid flow down an inclined plane. *Phys. Fluids* **6**, 321.